Graphical Expansion of Matrix Integrals with Values in a Clifford Algebra

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Abstract

The graphical expansions of non-commutative integrals over self-adjoint elements of certain types of Clifford Algebras are given in terms of power series over ribbon graphs and Möbius graphs. The coefficient of each term in the series depends only on the topological type of the ribbon or Möbius graph, and these formulae are computed explicitly in this paper.

Introduction

The asymptotic expansions of the matrix integrals over real symmetric, complex hermitian, and quaternionic self-adjoint matrices have been obtained using Feynman diagram expansions and ribbon graphs [3, 5]. Informally, this means that an integral over a class of matrices with entries in some algebra can be written as an infinite sum over graphs drawn on closed surfaces.

The results have been generalized to matrix integrals with values in group algebras over finite groups [6] and to those with values in von Neumann algebras, which are a more general class of non-commutative algebras [7]. Each matrix integral can be written as an infinite series where each term corresponds to a graph drawn over either an orientable or a non-orientable surface. What makes these expansions more interesting is that the coefficient of each term in the series depends only on the topological type of the corresponding graph, and hence may give us some information about topological properties of the surface on which the graph is drawn.

The most general result obtained is displayed in Theorem 1.1 from [7]. *(see page 48)*

Some examples of von Neumann algebras are the Clifford algebras, which have many applications, especially in physics. In the case of Clifford algebras $Cl_{n,0}$ with a positive definite quadratic form [2], the coefficients $A^{or}_{g,f}$ and $A^{nor}_{k,f}$ in the graphical expansion of the integrals can be formulated in simple explicit formulae involving only the dimension n of the underlying vector space for $Cl_{n,0}$. The purpose of this paper is to give those explicit formulae with the proofs. Our main result is displayed in Theorem 1.2, which was formulated using the same notations as in Theorem 1.1.

Theorem 1.1. *For a real von Neumann algebra A, we have*

(1.1)

$$\log \int_{\mathcal{H}_A} e^{-\frac{1}{4}\langle x^2\rangle} e^{\sum_{j=1}^{\infty} \frac{t_j}{2j}\langle x^j\rangle}\, d\mu(x) = \sum_{\substack{\Gamma \textit{ connected orientable} \\ \textit{Möbius graph}}} \frac{1}{|\mathrm{Aut}_M(\Gamma)|} A^{or}_{g(\Gamma),f(\Gamma)} \prod_j t_j^{v_j(\Gamma)}$$

$$+ \sum_{\substack{\Gamma \textit{ connected non-} \\ \textit{orientable Möbius graph}}} \frac{1}{|\mathrm{Aut}_M(\Gamma)|} A^{nor}_{k(\Gamma),f(\Gamma)} \prod_j t_j^{v_j(\Gamma)} .$$

where

$$d\mu(x) = \int_{\mathcal{H}_A} e^{-\frac{1}{4}\langle x^2\rangle}\, dx,$$

and $\mathcal{H}_A$ *is the set of self-adjoint elements of* A, $Aut_M(\Gamma)$ *is the automorphism group of the Möbius graph* Γ, S_Γ *is the Riemann surface associated to* Γ, $f(\Gamma)$ *is the number of faces of* Γ, $g(\Gamma)$ *is the genus of* S_Γ *if* S_Γ*is orientable, and* $k(\Gamma)$ *is the cross-cap genus of* S_Γ *if* S_Γ *is non-orientable;*

$$(1.2) \qquad A^{or}_{g,f} = \sum_{\substack{i_1,\ldots,i_g;j_1,\ldots,j_g \\ h_1,\ldots,h_{f-1}}}^{N} \left\langle e_{i_1} e_{j_1} e^*_{i_1} e^*_{j_1} \cdots e_{i_g} e_{j_g} e^*_{i_g} e^*_{j_g} \cdot e_{h_1} e^*_{h_1} \cdots e_{h_{f-1}} e^*_{h_{f-1}} \right\rangle,$$

$$(1.3) \qquad A^{nor}_{k,f} = \sum_{\substack{i_1,\ldots,i_k \\ h_1,\ldots,h_{f-1}}} \left\langle (e^*_{i_1})^2 \cdots (e^*_{i_k})^2 \cdot e_{h_1} e^*_{h_1} \cdots e_{h_{f-1}} e^*_{h_{f-1}} \right\rangle.$$

Theorem 1.2. *For a Clifford algebra* $A_n = Cl_{n,0}$.

(1.4)

$$\log \int_{\mathcal{H}_{A_n}} e^{-\frac{1}{4}\langle x^2\rangle} e^{\sum_j \frac{t_j}{2j}\langle x^j\rangle} d\mu(x)$$

$$= \sum_{\substack{\Gamma \textit{ connected orientable} \\ \textit{Möbius graph}}} \frac{1}{|Aut_M \Gamma|} \left(\frac{1}{2}\left(2^n - (-2)^n\right) + 2^n\right)^g (2^n)^{f(\Gamma)-1} \prod_j t_j^{v_j(\Gamma)}$$

$$+ \sum_{\substack{\Gamma \textit{ connected nonorientable} \\ \textit{Möbius graph}}} \frac{1}{|Aut_M \Gamma|} \left(\frac{1}{2}\left((1+\mathrm{i})^{n+1} + (1-\mathrm{i})^{n+1}\right)\right)^k (2^n)^{f(\Gamma)-1} \prod_j t_j^{v_j(\Gamma)}$$

Ribbon Graphs and Möbius Graphs

An *undirected graph* Γ is a 3-tuple (V, E, i), where V is called the set of vertices, E the set of edges, and $i : E \to (V x V)/\mathfrak{S}_2$ the incidence relation of the edges; i.e., $i(e) = \{u_1, u_2\}$ means that the edge e is incident to the vertices u_1 and u_2. A graph can be represented pictorially by a dot for each vertex and an arc connecting two vertices for every edge incident to those vertices.

A (traditional) *graph isomorphism* between two graphs is a pair of bijections between vertices and between edges that preserves the incidence relations. More precisely, for two graphs, $\Gamma_1 = (V_1, E_1, i_1)$, $\Gamma_2 = (V_2, E_2, i_2)$, a graph isomorphism $\varphi : \Gamma_1 \to \Gamma_2$ is a pair of bijections $\varphi_V : V_1 \to V_2$ and $\varphi_E : E_1 \to E_2$ such that if $i_1(e) = \{v_1, v_2\}$ in Γ_1, , then $i_2(\varphi_E(e)) = \{\varphi_V(v_1), \varphi_V(v_2)\}$ in Γ_2.

The *edge refinement* of a graph Γ is a new graph obtained by adding a new vertex on every edge, hence splitting each edge into two edges. More precisely, it is the graph $\Gamma' = (V', E', i')$, where the new vertex set V' is the disjoint union of the original vertices and the new vertices, one on each edge, so $V' = V \coprod V_E,\ V_E = \{u_e | e \in E\}.$ The new edge set E' is the disjoint union of two copies of the original edges, because there are two new edges for every original edge, so $E' = E \coprod E.$ The new incidence relation is $i' : E' \to V \times V_E.$, because every new edge is incident to an original vertex and a new vertex. We can then define the group $Aut(\Gamma)$ of automorphisms of a graph Γ to be the group of traditional graph isomorphisms from the edge of refinement of Γ to itself.

The edges in the edge refinement of Γ are called half-edges of Γ. The number of half-edges incident to a vertex is called the *valency* or *degree* of that vertex. At every vertex, we can define a cyclic ordering of half vertices incident to that vertex. An ordering of the vertices $u_1, \dots, u_j$ is in the form $(u_{k_1}, \dots, u_{k_j}),$, and in a cyclic ordering $(u_{k_1}, \dots, u_{k_j}),$ = $(u_{k_j}, u_{k_1}, \dots, u_{k_{j-1}})$. A graph with this cyclic structure at every vertex is called a *ribbon graph*. There are $(j - 1)!$ distinct cyclic orderings of half-edges at every j-valent vertex.

A Möbius graph is a ribbon graph with a $\mathfrak{S}_2$ coloring on every edge. A *vertex flip operation* at a vertex of a Möbius graph is an operation that reverses the cyclic order of the half-edges incident to the vertex and changes the $\mathfrak{S}_2$ coloring of every edge incident to the vertex. Two Möbius graphs obtained from one another by a finite sequence of vertex flip operations are identified.

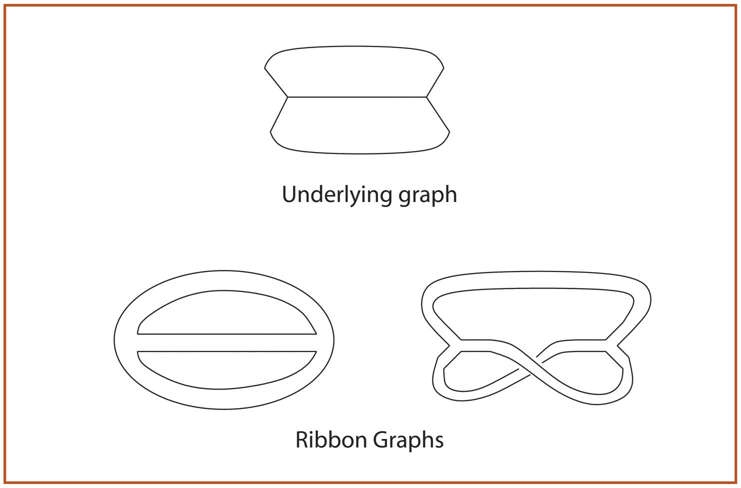


Figure 2.1. Two different ribbon graphs of the same underlying graph.

A *ribbon graph automorphism* $Aut_R(\Gamma)$ of a ribbon graph Γ is the subgroup of the automorphism group of the underlying graph that preserves the cyclic ordering of half-edges at every vertex in Γ. Similarly, a *Möbius graph automorphism* $Aut_M(\Gamma)$ of a Möbius graph Γ is the subgroup of the automorphic group of the underlying graph that preserves the cyclic orderings and the vertex flip operations on Γ.

Consider a ribbon graph $\Gamma = (V, E, i, c)$, where c is the cyclic ordering given to the vertices. As in Figure 2.2,

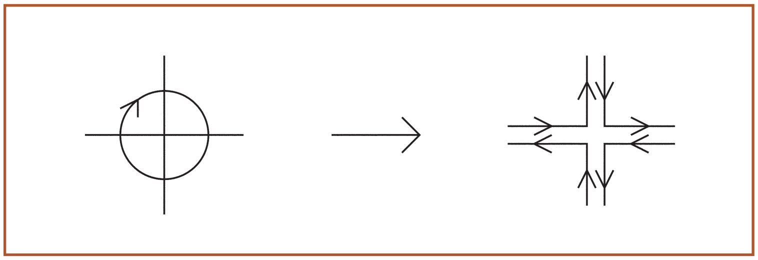
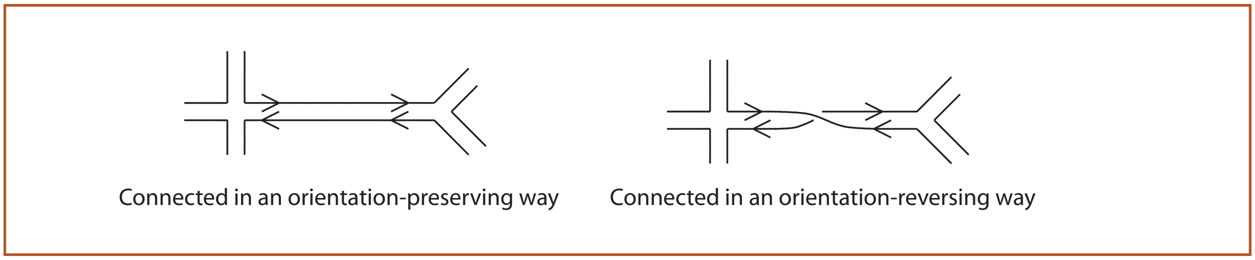


Figure 2.2. A vertex of a ribbon graph.

we can represent a half edge as a pair of arcs with opposite directions. These pairs of arcs come together at a *vertex* and are connected in a orientation-preserving manner as shown in the figure. The other end of a half edge, which is not connected to a vertex, is connected to another half edge. There are two ways to connect the arcs, orientation-preserving and orientation-reversing ways, as seen in Figure 2.3. For a ribbon graph, the half-edges must be connected in an orientation-preserving way. If we allow the half-edges to connect without regard for their orientation, we get a Möbius graph.



Connected in an orientation-preserving way

Connected in an orientation-reversing way

Figure 2.3. Two different ways to connect two half-edges.

We have realized the ribbon and Möbius graphs as oriented or non-orientable surfaces with boundaries. The boundary of a ribbon graph Γ is made of oriented closed curves. By attaching an oriented surface with a boundary to every "hole" made by one of the closed curves, we get an oriented compact surface S_Γ without a boundary. The boundary of a Möbius graph is made of oriented or non-orientable closed curves. By attaching an orientable or non-orientable surface to each boundary component, we get a compact Riemann surface S_Γ that is either orientable or non-orientable.

Each of the surfaces attached is called a face of the ribbon graph Γ. We denote by $f(\Gamma)$ the number of faces of Γ, $e(\Gamma) = |E|$, and $v(\Gamma) = |V|$. The Euler characteristic $\chi(S_\Gamma)$ of the surface S_Γ is defined as

$$\chi(S_\Gamma) = v(\Gamma) - e(\Gamma) + f(\Gamma).$$

If S_Γ is orientable, then

$$\chi(S_\Gamma) = 2 - 2g,$$

where g is the genus of the surface S_Γ. If S_Γ is non-orientable, then

$$\chi(S_\Gamma) = 2 - k,$$

where k is the cross-cap genus of S_Γ.

Feynman Diagrams

Let S be a finite set with even cardinality. A way of partitioning elements of S into disjoint pairs (of two elements each) is called a *pairing scheme* of that set. Let ~ be an arbitrary equivalence relation on S. Then we can consider the equivalence classes in $(S/\sim)$ as vertices of a graph and the pair $\{s_1, s_2\}$ as an edge incident to the equivalence classes $[s_1]$ and $[s_2]$. Hence, each pairing scheme of S gives rise to a graph. If we define cyclic ordering on each of the equivalence classes, then the graph becomes a ribbon graph, and each element of S corresponds to a half edge of the ribbon graph. See Figure 3.1.

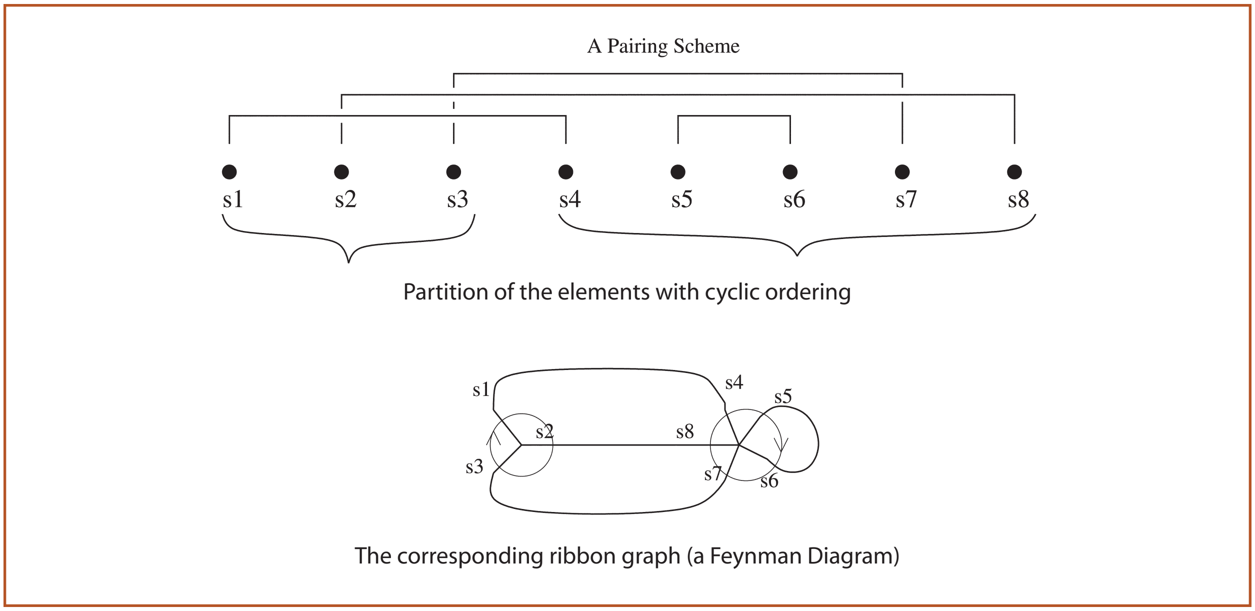


Figure 3.1. A pairing scheme and the corresponding Feynman Diagram.

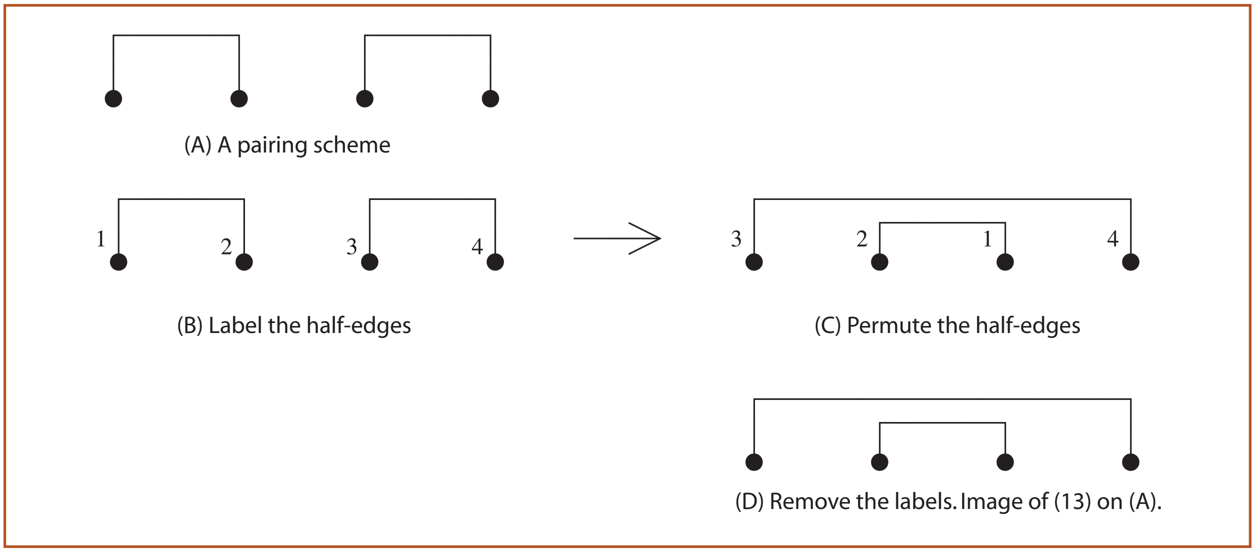


Figure 3.2. The action of $(13) \in \mathfrak{S}_4$ on a pairing scheme with one vertex.

Suppose $|S| = 2m$, and ~ partitions S into v_j subsets of j elements each, for $j = 1, 2, \ldots, 2m$. Then the semi-direct product $G = \prod_{j=1}^{2m} \mathfrak{S}_{v_j} \rtimes (\mathfrak{S}_j)^{v_j}$ acts on the set of pairing schemes S and X as follows. Each $\mathfrak{S}_{v_j}$ permutes the v_j equivalent classes in $S/\sim$ with j elements, and each $\mathfrak{S}_j$ permutes the elements in each equivalent class of j elements. Figure 3.2 shows the action of $(1\ 3) \in \mathfrak{S}_4$ on a pairing scheme with a single vertex. The action of permuting vertices is similar. Notice that this action does not change the graph corresponding to the pairing scheme. Hence, all the pairing schemes corresponding to the same graph have the same stabalizer subgroup under the action of G. Notice that the automorphism group of a graph Γ, coincides with the subgroup of $G = \prod_{j=1}^{2m} \mathfrak{S}_{v_j} \rtimes (\mathfrak{S}_j)^{v_j}$ that is the stabilizer of each pairing scheme corresponding to Γ. The orbit O_x of a pairing scheme x is the set of pairing schemes that correspond to the same graph as x, because one pairing scheme can be brought to another using a G-action if and only if these pairing schemes correspond to the same graph.

For a set X and a group action G on X, for every $x \in X$, we have

$$\text{(size of orbit) x (order of stabilizer) = (order of G)}$$

$$|O_x||G_x| = |G|$$

Hence,

$$\frac{|O_x|}{|G|} = \frac{1}{|G_x|},$$

and for a graph with v_j $j-$valent vertices for $j \geq 1$,

$$\frac{\#\text{ pairing schemes corresponding to }\Gamma}{\prod_{j\geq 1} v_j!(j!)^{v_j}} \qquad \frac{1}{|Aut(\Gamma)|}. \tag{3.1}$$

If we consider the action of $G = \prod_{j=1}^{2m} \mathfrak{S}_{v_j} \rtimes (\mathcal{C}_j)^{v_j}$ on X where $\mathcal{C}_j$ is the cyclic group of order j, then each orbit of G action on X is a ribbon graph because two pairing schemes can be brought from one to the other if and only if they correspond to the same graph and they have the same cyclic order of half-edges at every vertex. Then we have for a ribbon graph Γ with v_j $j-$valent vertices for $j \geq 1$,

$$\frac{\#\text{ pairing schemes corresponding to }\Gamma}{\prod_{j\geq 1} v_j! j^{v_j}} \qquad \frac{1}{|Aut_R(\Gamma)|}. \tag{3.2}$$

Now, consider the action of $G = \prod_{j=1}^{2m} \mathfrak{S}_{v_j} \rtimes (\mathcal{D}_j)^{v_j}$ on X where $\mathcal{D}_j$ is the dihedral group of order $2j$. Then each orbit of G action on X is a Möbius graph because a Möbius graph allows cyclic rotation and reflection (vertex flip operation) at every vertex. The following formula will be useful in later sections.

For a Möbius graph v_j $j-$valent vertices for $j \geq 1$, we have

$$\frac{\#\text{ pairing schemes corresponding to }\Gamma}{\prod_{j\geq 1} v_j!(2j)^{v_j}} \qquad \frac{1}{|Aut_M(\Gamma)|} \tag{3.3}$$

Clifford Algebras

Let V and W be two vector spaces over the same field F. Then their *tensor product* $V \bigotimes W$ is a vector space over F spanned by elements of the form $v \bigotimes w$ such that for each $v, v_1, v_2 \in V, w, w_1, w_2 \in W, c \in \mathbf{F}$.

(1) $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$
(2) $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$
(3) $c(v \otimes w) = (cv) \otimes w = v \otimes (cw)$.

This tensor product is associative.

Recall that an *algebra* is a vector space W over a field F with a multiplication which turns it into a ring, such that for any $c \in F, w_1, w_2 \in W$, we have $c(w_1 \cdot w_2) = (cw_1) \cdot w_2 = v_1 \cdot (cw_2)$. Let V be an n-dimensional vector space over a field F. For $r \geq 1$, let $T^r(V) = \bigotimes^r V = V \otimes \cdots \otimes V$ (r times), and $T^0(V) = F$. Then the direct sum $T(V) = \bigoplus_{r=0}^{\infty} T^r(V)$ has an algebraic structure, and it is called the tensor algebra of V over F.

Suppose q: $V \to F$ is a *quadratic form on* V. That is, given a basis B of V, there is an n x n matrix A with entries in F such that for each $v \in V, q(v) = [v]_{\mathcal{B}}^T A [v]_{\mathcal{B}}$, where $[v]_{\mathcal{B}}$ is the coordinate vector of v with respect to the basis B. Let $Iq(V)$ be the ideal in the tensor algebra $T(V)$ generated by $\{v \otimes v + q(v) | v \in V\}$. Then the Clifford algebra associated to V and q is defined to be the quotient

$$\mathcal{C}\ell(V, q) = T(V)/I_q(V). \tag{4.1}$$

In other words, $\mathcal{C}\ell(V, q)$ is the algebra obtained from $T(V)$ by setting

$$v \otimes v = -q(v)$$

for each $v \in V$. From now on, we will write $x \otimes y$ as xy when there is no confusion.

Now, consider the Clifford algebras $\mathcal{C}\ell_{r,s} = \mathcal{C}\ell(V, q)$, where $V = \mathbb{R}^{r+s}$ and $x = (x_1, \ldots, x_{r+s})$,

$$q(x) = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_{r+s}^2. \tag{4.2}$$

Here, we will consider only $Cl_{n,0}$. Hence, the matrix A mentioned earlier is the n x n identity matrix I_n.

Let $\{e_1, \ldots, e_n\}$ be the standard basis of $V = \mathbb{R}^n$. For any subset, $K = \{k_1 < \cdots < k_j\}$ of $[n] = \{1, \ldots, n\}$, let $e_K = e_{k_1} \cdots e_{k_j}$. We will use the following proposition without proof, which can be found in numerous books on the subject, including [2].

Proposition 4.1. *For the Clifford Algebra $\mathcal{C}\ell_{n,0}$, the set $\{e_K | \; K \subset [n]\}$ as defined above forms a basis.*

For any $1 \leq i, j \leq n, i \neq j$, we have $q(e_i + e_j) = 2$, and $q(e_i) = q(e_j) = 1$; hence

$$q(e_i + e_j) = -(e_i + e_j)(e_i + e_j) = -(e_i e_i + e_j e_j + e_i e_j + e_j e_i) = q(e_i) + q(e_j) - e_i e_j - e_j e_i$$

$$\implies 2 = 1 + 1 - e_i e_j - e_j e_i$$

$$\implies e_i e_j = -e_i e_j.$$

Moreover, $e_i^2 = -q(e_i) = -1$. Hence, we get

$$e_i e_j = \begin{cases} -e_j e_i & \textit{if } i \neq j \\ -1 & \textit{if } i = j \end{cases} \tag{4.3}$$

For $K = \{k_1 < \cdots < k_j\}$, consider $e_K^2 = (e_{k_1} \cdots e_{k_j})(e_{k_1} \cdots e_{k_j})$. The term e_{k_1} in the middle can change place with the e_{k_j} on its left with an extra factor of -1. Then it can again change place with the $e_{k_{j-1}}$, and so on. Hence, after changing place *j* - 1 times, the e_{k_1} is next to the leftmost e_{k_1}. Then these two terms vanish, leaving a factor -1, hence the total contribution of the two e_{k_1}'s is $(-1)^j$. Similarly, the two e_{k_2}'s vanish, leaving a factor $(-1)^{j-1}$. Continuing in this fashion, we get the formula

$$e_K^2 = (-1)^{\sum_{p=1}^{|K|} p} = (-1)^{\frac{(|K|+1)|K|}{2}}. \tag{4.4}$$

Now, for $K, L \subset [n]$, consider turning $e_K e_L = (e_{k_1} \cdots e_{k_j})(e_{l_1} \cdots e_{l_m})$ into $e_L e_K = (e_{l_1} \cdots e_{l_m})(e_{k_1} \cdots e_{k_j})$. Starting from e_{l_1}, each of the e_{l_α} can be brought to the new place by interchanging all of the j e_{k_β}'s. If $K \cap L$ is empty, then the number of (-1) factors obtained is $|K||L|$. However, for each element $p \in K \cap L$, interchanging two e_p's does not contribute anything. Thus the total contribution is $(-1)^{|K||L|-|K \cap L|}$, and $e_K e_L = (-1)^{|K||L|-|K \cap L|} e_L e_K$. Hence, we get

$$e_K e_L e_K^{-1} e_L^{-1} = (-1)^{|K||L|-|K \cap L|}. \tag{4.5}$$

Lemma 4.2. *For the basis* $\{e_K | \ K \subset [n]\}$ *of* $\mathcal{C}\ell_{n,0}$,

$$\sum_{K,L \subset [n]} e_K e_L e_K^{-1} e_L^{-1} = 2^n + \frac{1}{2}(2^n - (-2)^n) = \begin{cases} 2^n & \textit{if } n \textit{ is even} \\ 2^{n+1} & \textit{if } n \textit{ is odd} \end{cases},$$

Proof. From (4.5),

$$\sum_{(K,L) \in [n] \times [n]} e_K e_L e_K^{-1} e_L^{-1} = \sum_{(K,L) \in [n] \times [n]} (-1)^{|K||L|-|K \cap L|}. \tag{4.6}$$

We will now compute the right hand side of the above equation. Let $I \subset U \subset [n]$, $i = |I|$ and $j = |U \backslash I|$. We will consider pairs $(K, L) \in [n] \times [n]$ such that $K \cup L = U$ and $K \cap L = I$.

If *j* = 0, then *U* = *I*, so *K* = *L*. Hence, $|K||L| - |K \cap L| = |K|^2 - |K|$, which is always even, so $(-1)^{|K||L|-|K \cap L|} = (-1)^{|K|^2 - |K|} = 1$. Thus, every pair in [*n*] x [*n*] of the form (*K*, *K*) contributes 1 to the sum, so the total contribution of the pairs of the form (*K*, *K*) is 2^n.

There is a one-to-one correspondence between the pairs (*K*, *L*) such that $K \cap L = I, K \cup L = U$, and between the subsets of $U \setminus I$, because every subset $S \subset (U \backslash I)$ corresponds to a pair $(S \cup I, U \backslash S)$. There are 2^j subsets of $U \setminus I$, so that there are also 2^j pairs (*K*, *L*) such that $K \cap L = I, K \cup L = U$, .

Notice that for a non-empty set, the number of subets with even cardinalities is the same as the number of subsets with odd cardinalities. To see this, note that for any set *A* with ***|A| = n > 0,***

$$\begin{pmatrix} \text{number of subsets of A} \\ \text{with even cardinality} \end{pmatrix} - \begin{pmatrix} \text{number of subsets of A} \\ \text{with odd cardinality} \end{pmatrix}$$

$$= \sum_{B \subset A} (-1)^{|B|} = \sum_{i=0}^{n} \binom{n}{i} (-1)^i = (1-1)^n = 0.$$

Now, suppose J is positive and odd. Then,

$$|K|+|L|=2|K\cap L|+|(K\cup L)\backslash(K\cap L)|=2i+j$$

is odd, and one in the pair (K, L) is odd and the other is even. Thus, $|K|\,|L|$ is even, and

$$(-1)^{|K||L|-|K\cap L|}=(-1)^{-|K\cup L|}=(-1)^{-i}=(-1)^{i}.$$

Hence, in this case, each pair (K, L) contributes $(-1)^i$ to the sum on the right hand side of (4.6). Therefore, for odd $j > 0$, the total contribution from any I and U is $(-1)^i2^j$.

Now suppose $j > 0$ is even. Then $|K| + |L| = 2i + j$ is even, so $|K|$ and $|L|$ are either both odd or both even. Hence, $|K||L|$ is even when $|K|$ is even, and odd when $|K|$ is odd. As seen above, exactly half of the subsets of U/I have even cardinalities and the other half have odd cardinalities. Therefore, among the pairs (K,L) where $K\cap L=I$ and $K\cup L=U$, $|K|$ is even half of the times and odd half of the times, and so is $|K||L|$ Hence, $(-1)^{|K||L|-|K\cap L|}$ is equal to 1 in one-half of the cases, and -1 in the other half, so the total contribution to the sum is 0.

Therefore,

$$\sum_{K,L\subset[n]}(-1)^{|K||L|-|K\cap L|}=2^n+\sum_{i=0}^{n}\sum_{j=1}^{n-i}\binom{n}{i}\binom{n-i}{j}f(i,j),$$

where $f(i,j)=\begin{cases}(-1)^i2^j & \text{if } j \text{ is odd}\\ 0 & \text{if } j \text{ is even}\end{cases}=\frac{1}{2}(-1)^i(2^j-(-2)^j)$. In this formula, $f(i,j)=0$ when $j=0$, so we can change the summation to make j start from 0 instead of 1. Then

$$\begin{aligned}\sum_{K,L\subset[n]}(-1)^{|K||L|-|K\cap L|}&=2^n+\sum_{i=0}^{n}\sum_{j=1}^{n-i}\binom{n}{i}\binom{n-i}{j}\frac{1}{2}(-1)^i(2^j-(-2)^j)\\&=2^n+\frac{1}{2}\left(\sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}(-1)^i(2^j-(-2)^j)\right)\\&=2^n+\frac{1}{2}\sum_{i=0}^{n}\binom{n}{i}(-1)^i\left(\sum_{j=0}^{n-i}\binom{n-i}{j}2^j-\sum_{j=0}^{n-i}\binom{n-i}{j}(-2)^j\right)\\&=2^n+\frac{1}{2}\sum_{i=0}^{n}\binom{n}{i}(-1)^i\left((2+1)^{n-i}-(-2+1)^{n-i}\right)\\&=2^n+\frac{1}{2}\left((-1+3)^n-(-1-1)^n\right)\\&=2^n+\frac{1}{2}\left(2^n-(-2)^n\right)\\&=\begin{cases}2^n & \textit{if } n \textit{ is even}\\ 2^{n+1} & \textit{if } n \textit{ is odd}\end{cases}\end{aligned}$$

Lemma 4.3. *For the basis* $\{e_K \mid K \subset [n]\}$ *of* $\mathcal{C}\ell_{n,0}$,

$$\sum_{K\subset[n]} e_K^2 = \frac{1}{2}\left((1+\mathrm{i})^{n+1} + (1-\mathrm{i})^{n+1}\right),$$

where $\mathrm{i} = \sqrt{-1}$.

Proof. From (4.4),

$$e_K^2 = (-1)^{\frac{(|K|+1)|K|}{2}} = \begin{cases} 1 & \textit{if } |K| = 0, 3 (\textit{mod } 4) \\ -1 & \textit{if } |K| = 1, 2 (\textit{mod } 4) \end{cases}.$$

Hence, $\sum_{K\subset[n]} e_K^2 = s_n - a_n$ where s_n is the number of subsets of $[n]$ whose cardinality is 0 or 3 (mod 4), and a_n is the number of subsets of $[n]$ whose cardinality is 1 or 2 (mod 4). Thus,

$$s_n = \sum_{\substack{0\le k\le n \\ k=0,3(mod\ 4)}} \binom{n}{k},$$

$$a_n = \sum_{\substack{0\le k\le n \\ k=1,2(mod\ 4)}} \binom{n}{k}.$$

Using the basic formula

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k},$$

we get

$$\begin{aligned}
Re\left((1+\mathrm{i})^{n+1}\right) &= Re\left(\sum_{k=0}^{n+1} \binom{n+1}{k} \mathrm{i}^k\right) \\
&= Re\left(\sum_{k=0}^{n+1} \left(\binom{n}{k-1} + \binom{n}{k}\right) \mathrm{i}^k\right) \\
&= \sum_{\substack{0\le k\le n+1 \\ k=0(mod\ 4)}} \left(\binom{n}{k-1} + \binom{n}{k}\right) - \sum_{\substack{0\le k\le n+1 \\ k=2(mod\ 4)}} \left(\binom{n}{k-1} + \binom{n}{k}\right) \\
&= \sum_{\substack{0\le k\le n \\ k=3(mod\ 4)}} \binom{n}{k} + \sum_{\substack{0\le k\le n \\ k=0(mod\ 4)}} \binom{n}{k} - \sum_{\substack{0\le k\le n \\ k=1(mod\ 4)}} \binom{n}{k} - \sum_{\substack{0\le k\le n \\ k=2(mod\ 4)}} \binom{n}{k} \\
&= \sum_{\substack{0\le k\le n \\ k=0,3(mod\ 4)}} \binom{n}{k} - \sum_{\substack{0\le k\le n \\ k=1,2(mod\ 4)}} \binom{n}{k} \\
&= s_n - a_n \\
&= \sum_{K\subset[n]} e_K^2
\end{aligned}$$

Integrals over Clifford Algebras: Proof of the Main Theorem

Let $A_n = \mathcal{C}\ell_{n,0}$ be a Clifford algebra with the basis $\{e_K | K \subseteq [n]\}$. Recall that $e_K^2 = \pm 1$ and that $e_K^4 = 1$. Any element $y \in A_n$ has the form

$$y = \sum_{K \subseteq [n]} y^K e_K,$$

where each $y^K \in \mathbb{R}$. Define the adjoint operator by

$$y^* = \sum_{K \subseteq [n]} y^K e_K^{-1} = \sum_{K \subseteq [n]} y^K e_K^2 e_K.$$

Then $(y^*)^* = y$ and $y + y^*$ is self-adjoint for any e_{k_2}'s If $x \in A_n$ is self-adjoint, then

$$\sum_{K \subseteq [n]} x^K e_K = x = x^* = \sum_{K \subseteq [n]} x^K e_K^{-1} = \sum_{K \subseteq [n]} x^K e_K^2 e_K,$$

so $x^K = e_K^2 x^K$ for all $K \subseteq [n]$. Define the *trace* $\langle \cdot \rangle : A_n \to \mathbb{R}$ by

$$\left\langle \sum_{K \subseteq [n]} y^K e_K \right\rangle = y^{\phi}.$$

Let $\mathcal{H}_{A_n}$ denote the space of all self-adjoint elements of A_n, and define the differential operator $\frac{\partial}{\partial y}$ on A_n by

$$\frac{\partial}{\partial y} = \sum_{K \subseteq [n]} \frac{\partial}{\partial y^K} e_K^{-1}.$$

Lemma 5.1. *If $x \in A_n$ is self-adjoint, then*

$$\frac{\partial}{\partial y} e^{\frac{1}{2}\langle x(y+y^*)\rangle} = x e^{\frac{1}{2}\langle x(y+y^*)\rangle}. \tag{5.1}$$

Proof. Since x is self-adjoint, $x^K = e_K^2 x^K$ for all $K \subseteq [n]$.

$$\begin{aligned}
\frac{\partial}{\partial y} e^{\frac{1}{2}\langle x(y+y^*)\rangle} &= \left(\sum_{K \subseteq [n]} \frac{\partial}{\partial y} e_K^{-1} \right) \left(e^{\frac{1}{2}\sum_{K \subseteq [n]}(x^K y^K + x^K y^K e_K^2)} \right) \\
&= \frac{1}{2} \sum_{K \subseteq [n]} (x^K + x^K e_K^2) e_K^{-1} e^{\frac{1}{2}\langle x(y+y^*)\rangle} \\
&= \frac{1}{2} \sum_{K \subseteq [n]} (2x^K) e_K^{-1} e^{\frac{1}{2}\langle x(y+y^*)\rangle} \\
&= x e^{\frac{1}{2}\langle x(y+y^*)\rangle}.
\end{aligned}$$

Lemma 5.2. *If $x \in A_n$ is self-adjoint, then*

$$\left\langle \left(\frac{\partial}{\partial y}\right)^j \right\rangle^l e^{\frac{1}{2}\langle x(y+y^*)\rangle} = \langle x^j \rangle^l e^{\frac{1}{2}\langle x(y+y^*)\rangle} \tag{5.2}$$

Proof. Note that $e^{\frac{1}{2}\langle x(y+y^*)\rangle} \in \mathbb{R}$. Thus, using (5.1) *j* times gives

$$\begin{aligned} \left\langle \left(\frac{\partial}{\partial y}\right)^j \right\rangle e^{\frac{1}{2}\langle x(y+y^*)\rangle} &= \left\langle \left(\frac{\partial}{\partial y}\right)^j e^{\frac{1}{2}\langle x(y+y^*)\rangle} \right\rangle \\ &= \left\langle x^j e^{\frac{1}{2}\langle x(y+y^*)\rangle} \right\rangle \\ &= \left\langle x^j \right\rangle e^{\frac{1}{2}\langle x(y+y^*)\rangle}. \end{aligned} \tag{5.3}$$

Repeating (5.3) *l* times gives (5.2).

From the above lemma, we get

$$\left\langle \left(\frac{\partial}{\partial y}\right)^j \right\rangle^l e^{\frac{1}{2}\langle x(y+y^*)\rangle} \Bigg|_{y=0} = \langle x^j \rangle^l$$

We will now go back to the formula in Theorem 1.2.

Proof. Proof of Theorem 1.2.

From the Taylor expansion of the exponent function, we get

$$\begin{aligned} &\int_{\mathcal{H}_{A_n}} e^{-\frac{1}{4}\langle x^2\rangle} e^{\sum_{j=1}^{\infty} \frac{t_j}{2j}\langle x^j\rangle} d\mu(x) \\ &= \int_{\mathcal{H}_{A_n}} e^{-\frac{1}{4}\langle x^2\rangle} \prod_{j=1}^{\infty} e^{\frac{t_j}{2j}\langle x^j\rangle} d\mu(x) \\ &= \int_{\mathcal{H}_{A_n}} e^{-\frac{1}{4}\langle x^2\rangle} \prod_{j=1}^{\infty} \left(\sum_{v_j=0}^{\infty} \frac{1}{v_j!} \left(\frac{t_j}{2j} \langle x^j \rangle \right)^{v_j} \right) d\mu(x) \\ &= \int_{\mathcal{H}_{A_n}} e^{-\frac{1}{4}\langle x^2\rangle} \sum_{(v_1,v_2,v_3,\dots)\in\mathbb{N}^\infty} \frac{t_1^{v_1}\ t_2^{v_2}\ t_3^{v_3}\ \cdots}{v_1!\ v_2!\ v_3!\ \cdots\ 2^{v_1}\ 4^{v_2}\ 6^{v_3}\ \cdots} \langle x^1\rangle^{v_1} \langle x^2\rangle^{v_2} \langle x^3\rangle^{v_3} \cdots\ d\mu(x) \\ &= \sum_{(v_1,v_2,v_3,\dots)\in\mathbb{N}^\infty} \prod_{j=0}^{\infty} \frac{t_j^{v_j}}{v_j!\ (2j)^{v_j}} \int_{\mathcal{H}_{A_n}} e^{-\frac{1}{4}\langle x^2\rangle} \prod_{j=0}^{\infty} \langle x^j\rangle^{v_j} d\mu(x). \end{aligned}$$

Now we will compute $\int_{\mathcal{H}_{A_n}} e^{-\frac{1}{4}\langle x^2\rangle} \prod_{j=0}^{\infty} \langle x^j\rangle^{v_j} d\mu(x)$. Recall that $\int_{\mathcal{H}} e^{-\frac{1}{4}\langle x^2\rangle} d\mu(x) = 1$, and that

$$\int_{\mathcal{H}} e^{-\frac{1}{4}\langle (x-z)^2\rangle} d\mu(x) = 1 \tag{5.4}$$

after the coordinate change $x \to x - z$. Using (5.4) and (5.2),

$$\begin{aligned}
\int_{\mathcal{H}_{A_n}} e^{-\frac{1}{4}\langle x^2\rangle} \prod_{j=0}^{\infty} \langle x^j\rangle^{v_j}\, d\mu(x) &= \int_{\mathcal{H}_{A_n}} e^{-\frac{1}{4}\langle x^2\rangle} \prod_{j=1}^{\infty} \left\langle \left(\frac{\partial}{\partial y}\right)^j \right\rangle^{v_j} e^{\frac{1}{2}\langle x(y+y^*)\rangle}\Big|_{y=0} d\mu(x) \\
&= \prod_{j=1}^{\infty} \left\langle \left(\frac{\partial}{\partial y}\right)^j \right\rangle^{v_j} \int_{\mathcal{H}_{A_n}} e^{-\frac{1}{4}\langle x^2\rangle} e^{\frac{1}{2}\langle x(y+y^*)\rangle}\, d\mu(x)\Big|_{y=0} \\
&= \prod_{j=1}^{\infty} \left\langle \left(\frac{\partial}{\partial y}\right)^j \right\rangle^{v_j} \int_{\mathcal{H}_{A_n}} e^{-\frac{1}{4}\langle (x-(y+y^*))^2\rangle}\, d\mu(x)\; e^{\frac{1}{4}\langle (y+y^*)^2\rangle}\Big|_{y=0} \\
&= \prod_{j=1}^{\infty} \left\langle \left(\frac{\partial}{\partial y}\right)^j \right\rangle^{v_j} e^{\frac{1}{4}\langle (y+y^*)^2\rangle}\Big|_{y=0}
\end{aligned}$$

Thus, the integral we are computing becomes

$$\sum_{(v_1,v_2,v_3,\dots)\in\mathbb{N}^\infty} \prod_{j=1}^{\infty} \left\langle \left(\frac{\partial}{\partial y}\right)^j \right\rangle^{v_j} e^{\frac{1}{4}\langle (y+y^*)^2\rangle}\Big|_{y=0} \prod_{j=0}^{\infty} \frac{t_j^{v_j}}{v_j!\,(2j)^{v_j}}. \tag{5.5}$$

By straightforward computation, using the properties of the e_K's, we get

$$\begin{aligned}
\frac{\partial}{\partial y^K} e^{\frac{1}{4}\left\langle (y+y^*)^2\right\rangle} &= \frac{\partial}{\partial y^K} e^{\frac{1}{4}\left\langle \left(\sum_{M\subset[n]}(1+e_M^2) y^M e_M\right)^2\right\rangle} \\
&= \frac{\partial}{\partial y^K} e^{\frac{1}{4}\sum_{M\subset[n]}(1+e_M^2)^2 (y^M e_M)^2} \\
&= \frac{1}{2} y^K e_K^2 (1+e_K^2)^2 e^{\frac{1}{4}\sum_{M\subset[n]}(1+e_M^2)^2 (y^M e_M)^2} \\
&= \frac{1}{2} y^K e_K^2 (2+2e_K^2) e^{\frac{1}{4}\left\langle (y+y^*)^2\right\rangle} \\
&= y^K e_K^2 (1+e_K^2) e^{\frac{1}{4}\left\langle (y+y^*)^2\right\rangle} \\
&= \left(y^K + y^K e_K^2\right) e^{\frac{1}{4}\left\langle (y+y^*)^2\right\rangle}.
\end{aligned} \tag{5.6}$$

Hence

$$\begin{aligned}
\frac{\partial}{\partial y^L}\frac{\partial}{\partial y^K} e^{\frac{1}{4}\left\langle (y+y^*)^2\right\rangle}\Big|_{y=0} &= \frac{\partial}{\partial y^L}\left(y^K + y^K e_K^2\right) e^{\frac{1}{4}\left\langle (y+y^*)^2\right\rangle}\Big|_{y=0} \\
&= \delta_{KL} + \delta_{KL} e_K^2 \\
&= \langle e_K e_L^{-1}\rangle + \langle e_K e_L\rangle.
\end{aligned} \tag{5.7}$$

Let us rewrite equation (5.5) as

$$\sum_{n=0}^{\infty} \sum_{\substack{(v_1,v_2,v_3,\dots)\in\mathbb{N}^\infty \\ \mathbf{e}(v_1,v_2,v_3,\dots)=n}} \prod_{j=1}^{\text{finite}} \frac{1}{v_j!\,(2j)^{v_j}} \left\langle \left(\frac{\partial}{\partial y}\right)^j \right\rangle^{v_j} e^{\frac{1}{4}\langle (y+y^*)^2\rangle}\Big|_{y=0} t_j^{v_j}. \tag{5.8}$$

where $\mathbf{e}(v_1, v_2, v_3, \dots) = \sum_{j\geq 1} j v_j$. The product is finite because for finite $\mathbf{e}(v_1, v_2, v_3, \dots)$, there are only a finite number of non-zero v_j's.

By the definition of the differential operator,

$$\left\langle \left(\frac{\partial}{\partial y}\right)^{j} \right\rangle = \sum_{K_1,\ldots,K_j} \frac{\partial}{\partial y^{K_1}} \cdots \frac{\partial}{\partial y^{K_j}} \langle e_{K_1}^{-1} \cdots e_{K_j}^{-1} \rangle. \tag{5.9}$$

Using (5.9), each summand in (5.8) contains $\sum_{j\geq 1} j v_j$ differential operators. This number is also $\mathbf{e}(v_1, v_2, v_3, \ldots)$. Notice from (5.6) and (5.7) that all the differential operators must be paired to get a non-zero contribution. As shown in the section on Feynman Diagrams, each pairing scheme gives a graph. Since the trace is invariant under the cyclic ordering, each pairing scheme gives a ribbon graph. From (5.7), every pair of differential operators receives a contribution $\langle e_K e_L^{-1} \rangle + \langle e_K e_L \rangle$. Also notice that there are two ways to connect two half-edges (Fig. 2.3). Thus, we will say that an edge has a contribution $\langle e_K e_L \rangle$ if it is connected in an orientation-preserving way and has a contribution $\langle e_K e_L^{-1} \rangle$ if it is connected in an orientation-reversing way. Therefore, both orientable and non-orientable Möbius graphs arise. There is a natural way to partition the half-edges (the differential operators) according to the term $\langle e_{K_1}^{-1} \cdots e_{K_j}^{-1} \rangle$, hence each vertex has the contribution $\langle e_{K_1}^{-1} \cdots e_{K_j}^{-1} \rangle$, Also note that there are v_j $j-$valent vertices in the corresponding Möbius graph. Using (3.3), we get

$$\int_{\mathcal{H}_{A_n}} e^{-\frac{1}{4}\langle x^2 \rangle} e^{\sum_j \frac{t_j}{2j} \langle x^j \rangle} d\mu(x) = \sum_{\Gamma \text{ Möbius Graph}} \frac{1}{|\mathrm{Aut}_M(\Gamma)|} A_\Gamma$$

where $Aut_M(\Gamma)$ is the automorphism group of Γ as a Möbius graph and A_Γ is the contribution made by Γ. This A_Γ is equal to the sum of contributions of all possible ways of assigning e_K to each half vertex, so that the product is $\langle e_{K_1}^{-1} \cdots e_{K_j}^{-1} \rangle = \pm 1$ at each *j*-valence vertex, $\langle e_K e_L^{-1} \rangle = \pm 1$ for every twisted edge, and $\langle e_K e_L \rangle = \pm 1$ for every straight edge. The only way either $\langle e_K e_L \rangle$ or $\langle e_K e_L^{-1} \rangle$ can be non-zero is when *K* = *L*, so a half- edge, labeled e_K, can only be connected to another half edge, labeled e_K, contributing $\langle e_K e_K^{-1} \rangle$, with a twisted edge or $\langle e_K e_K \rangle$. with a straight edge.

This Feynman diagram expansion formula is summed over all Möbius graphs, both connected and not connected. The graphical expansion of the formal logarithm of the integral involves only the connected Möbius graphs as shown in [3]. Hence,

$$\log \int_{\mathcal{H}_{A_n}} e^{-\frac{1}{4}\langle x^2 \rangle} e^{\sum_j \frac{t_j}{2j} \langle x^j \rangle} d\mu(x) = \sum_{\Gamma \text{ Connected Möbius Graph}} \frac{1}{|\mathrm{Aut}_M(\Gamma)|} A_\Gamma$$

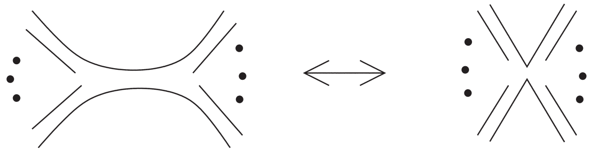


Figure 5.1. Contraction and expansion of an edge.

Consider *contraction* and *expansion* of edges (as shown in Fig. 5.1). Suppose v_1 and v_2 are two vertices of a Möbius graph Γ, and there is an edge *e* between them. The contribution from v_1 is $\langle a e_K^{-1} \rangle$ and that from v_2 is $\langle e_L^{-1} b \rangle$. Contracting the edge *e* changes $\sum_{K,L\in[n]} \langle a e_K^{-1} \rangle \langle e_K e_L \rangle \langle e_L^{-1} b \rangle$ to (*ab*). Since the product is $e_{K_1} \cdots e_{K_i} = \pm e_M$ for some *M*, we can write $a = p e_M, b = q e_N$ where $p, q = \pm 1$. We can then see that

$$
\begin{aligned}
\sum_{K,L\in[n]} \left\langle ae_K^{-1}\right\rangle \left\langle e_K e_L\right\rangle \left\langle e_L^{-1}b\right\rangle &= \sum_{K\in[n]} \left\langle ae_K^2 e_K\right\rangle e_K^2 \left\langle e_K^2 e_K b\right\rangle \\
&= \sum_{K\in[n]} \left\langle ae_K\right\rangle e_K^2 \left\langle e_K b\right\rangle \\
&= \sum_{K\in[n]} \left\langle pe_M e_K\right\rangle e_K^2 \left\langle e_K pe_M\right\rangle .
\end{aligned}
\tag{5.10}
$$

If $M \neq N$, then the above formula is equal to 0 and $\langle ab\rangle = \langle pe_M qe_N\rangle = 0$, so

$$
\sum_{K,L\in[n]} \left\langle ae_K^{-1}\right\rangle \left\langle e_K e_L\right\rangle \left\langle e_L^{-1}b\right\rangle = \langle ab\rangle . \tag{5.11}
$$

If $\mathbf{M = N}$, then the formula in (5.10) becomes

$$
\langle pe_M e_M\rangle e_M^2 \langle e_M qe_M\rangle = pe_M e_M e_M^2 e_M qe_M = pe_M qe_M = \langle ab\rangle ,
$$

so (5.11) holds true again.

Now let us consider orientable Möbius graphs, i.e., those Möbius graphs whose edges are connected in orientation-preserving ways. By contracting and expanding edges as in Figure 5.1, the Möbius graph can be brought to the ribbon graph in Figure 5.2. The above computation shows that contracting or expanding edges does not change the value of A_Γ .

Therefore, the contribution A_Γ depends only on its topological type $(f(\Gamma)$ and $g(S_\Gamma)$, and using $e_K^{-1}e_K^2 = e_K^4 e_K = e_K$,

$$
\begin{aligned}
A^{\mathrm{or}}_{g,f} &= \sum_{\substack{K_1,\ldots,K_g,L_1,\ldots,L_g,\\ S_1,\ldots,S_g,T_1,\ldots,T_g,\\ M_1,\ldots,M_{f-1};N_1,\ldots,N_{f-1}}} \begin{aligned}[t] &\left\langle e_{S_1}^{-1}e_{T_1}^{-1}e_{K_1}^{-1}e_{L_1}^{-1}\cdots e_{S_g}^{-1}e_{T_g}^{-1}e_{K_g}^{-1}e_{L_g}^{-1}e_{M_1}^{-1}e_{N_1}^{-1}\cdots e_{M_{f-1}}^{-1}e_{N_{f-1}}^{-1}\right\rangle \\ &\times \left\langle e_{K_1}e_{S_1}\right\rangle \left\langle e_{L_1}e_{T_1}\right\rangle \cdots \left\langle e_{K_g}e_{S_g}\right\rangle \left\langle e_{L_g}e_{T_g}\right\rangle \left\langle e_{M_1}e_{N_1}\right\rangle \cdots \left\langle e_{M_{f-1}}e_{N_{f-1}}\right\rangle \end{aligned} \\
&= \sum_{\substack{K_1,\ldots,K_g,L_1,\ldots,L_g,\\ M_1,\ldots,M_{f-1}}} \begin{aligned}[t] &\left\langle e_{K_1}^{-1}e_{L_1}^{-1}e_{K_1}^{-1}e_{L_1}^{-1}\cdots e_{K_g}^{-1}e_{L_g}^{-1}e_{K_g}^{-1}e_{L_g}^{-1}\cdot e_{M_1}^{-1}e_{M_1}^{-1}\cdots e_{M_{f-1}}^{-1}e_{M_{f-1}}^{-1}\right\rangle \\ &\times e_{K_1}^2 e_{L_1}^2\cdots e_{K_g}^2 e_{L_g}^2 e_{M_1}^2\cdots e_{M_{f-1}}^2 \end{aligned} \\
&= \sum_{\substack{K_1,\ldots,K_g,L_1,\ldots,L_g,\\ M_1,\ldots,M_{f-1}}} \left\langle e_{K_1}e_{L_1}e_{K_1}^{-1}e_{L_1}^{-1}\cdots e_{K_g}e_{L_g}e_{K_g}^{-1}e_{L_g}^{-1}\cdot e_{M_1}e_{M_1}^{-1}\cdots e_{M_{f-1}}e_{M_{f-1}}^{-1}\right\rangle . \\
&= \sum_{\substack{K_1,\ldots,K_g,L_1,\ldots,L_g,\\ M_1,\ldots,M_{f-1}}} \left\langle e_{K_1}e_{L_1}e_{K_1}^{-1}e_{L_1}^{-1}\cdots e_{K_g}e_{L_g}e_{K_g}^{-1}e_{L_g}^{-1}\right\rangle .
\end{aligned}
$$

Since the indices $M_1, \ldots, M_{f-1}$ can be any subset of $[n]$, there are 2^N choices for each M_i. Combining this with Lemma 4.2, we get

$$
A^{\mathrm{or}}_{g,f} = \left(\frac{1}{2}\left(2^n - (-2)^n\right) + 2^n\right)^g \left(2^n\right)^{f-1}. \tag{5.12}
$$

If Γ is not orientable, then it can be brought to the standard non-orientable graph by a sequence of vertex flip operations, edge contractions and expansions.

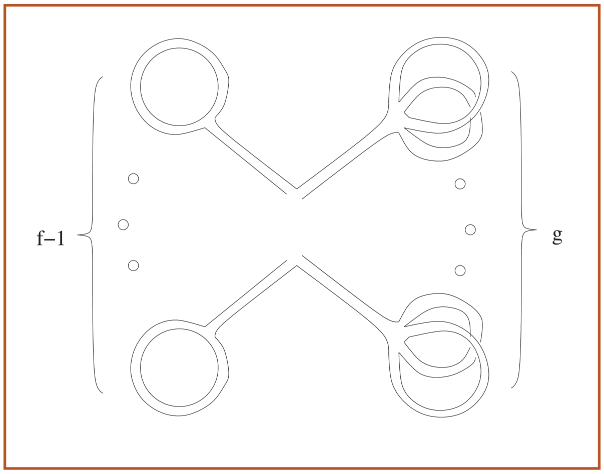


Figure 5.2. Standard graph for orientable (ribbon) graphs.

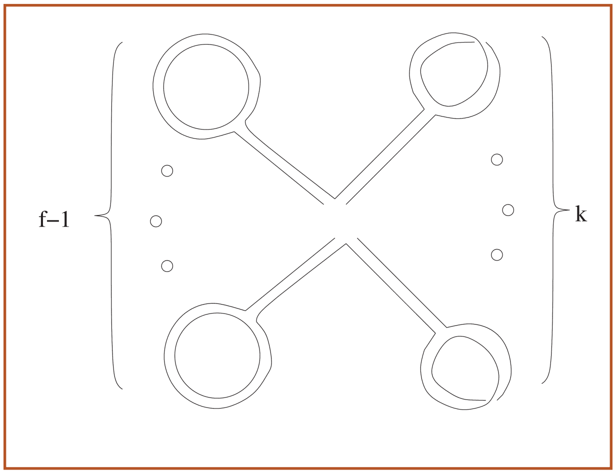


Figure 5.3. Standard graph for non-orientable Möbius graphs.

From the standard graph, we can see that the contribution is

$$\begin{aligned}
A^{\text{nor}}_{k,f} &= \sum_{\substack{K_1,\dots,K_k;L_1,\dots,L_k;\\ M_1,\dots,M_{f-1};N_1,\dots,N_{f-1}}} \begin{array}{l} \left\langle e^{-1}_{K_1}e^{-1}_{L_1}\cdots e^{-1}_{K_k}e^{-1}_{L_k}e^{-1}_{M_1}e^{-1}_{N_1}\cdots e^{-1}_{M_{f-1}}e^{-1}_{N_{f-1}}\right\rangle \\ \times \left\langle e_{K_1}e^{-1}_{L_1}\right\rangle\cdots\left\langle e_{K_k}e^{-1}_{L_k}\right\rangle \langle e_{M_1}e_{N_1}\rangle\cdots\langle e_{M_{f-1}}e_{N_{f-1}}\rangle \end{array} \\
&= \sum_{\substack{K_1,\dots,K_k;\\ M_1,\dots,M_{f-1}}} \left\langle (e^{-1}_{K_1})^2\cdots(e^{-1}_{K_k})^2(e^{-1}_{M_1})^2\cdots(e^{-1}_{M_{f-1}})^2\right\rangle \; e^2_{M_1}\cdots e^2_{M_{f-1}} \\
&= \sum_{\substack{K_1,\dots,K_k;\\ M_1,\dots,M_{f-1}}} (e^2_{K_1}e_{K_1})^2\cdots(e^2_{K_k}e_{K_k})^2(e^2_{M_1}e_{M_1})^2\cdots(e^2_{M_{f-1}}e_{M_{f-1}})^2 \; e^2_{M_1}\cdots e^2_{M_{f-1}} \\
&= \sum_{\substack{K_1,\dots,K_k;\\ M_1,\dots,M_{f-1}}} (e_{K_1})^2\cdots(e_{K_k})^2
\end{aligned}$$

Since the indices $M_1,\dots,M_{f-1}$ can be any subset of $[n]$, there are 2^N choices for each M_i. From this and Lemma 4.3, we get

$$A^{\text{nor}}_{k,f} = \left(\frac{1}{2}\left((1+i)^{n+1}+(1-i)^{n+1}\right)\right)^k (2^n)^{f(\Gamma)-1}. \tag{5.13}$$

From (5.12) and (5.13), it follows that

(5.14)

$$\begin{aligned}
&\log \int_{\mathcal{H}_{A_n}} e^{-\frac{1}{4}\langle x^2\rangle} e^{\sum_j \frac{t_j}{2j}\langle x^j\rangle} d\mu(x)\\
&\quad = \sum_{\substack{\Gamma \text{ connected orientable}\\ \text{Möbius graph}}} \frac{1}{|Aut\Gamma|}\left(\frac{1}{2}\left(2^n-(-2)^n\right)+2^n\right)^g (2^n)^{f(\Gamma)-1}\prod_j t_j^{v_j(\Gamma)}\\
&\quad + \sum_{\substack{\Gamma \text{ connected nonorientable}\\ \text{Möbius graph}}} \frac{1}{|Aut\Gamma|}\left(\frac{1}{2}\left((1+i)^{n+1}+(1-i)^{n+1}\right)\right)^k (2^n)^{f(\Gamma)-1}\prod_j t_j^{v_j(\Gamma)}.
\end{aligned}$$

Such explicit formulae are not yet known for other Clifford Algebras $\mathcal{C}\ell_{r,s}$ or for the general $\mathcal{C}\ell(V,q)$.

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